# THE BOUNDARY LAYER ON A PLATE THE SURFACE TEMPERATURE OF WHICH VARIES IN TIME

## (POGRANICHNYI SLOI NA PLASTINE S IZMENIAIUSHCHEISIA VO VREMENI TEMPERATUROI POVERKHNOSTI)

*PMM Vol.24, No.4, 1960, pp. 647-650*

Iu.A. DEM'IANOV
(Moscow)

*(Received 10 October 1960)*

We assume that a semi-infinite plate, set in a uniform flow of fluid of velocity $V_\infty$ has started to heat up according to a law $T_w(t)$ ($T_{w0}$ is the initial temperature of the surface). The problem consists in determining the temperature distribution in the boundary layer.

Assuming the kinematic viscosity $\nu$ constant over the velocity field $v_x$ and $v_y$, we have the Blasius solution for the boundary layer as follows:

$$v_x = V_\infty f'(\zeta), \qquad v_y = {}^1/_2 \sqrt{\frac{\nu V_\infty}{x}}(\zeta f' - f), \qquad \zeta = \frac{y}{\sqrt{\nu x / V_\infty}} \tag{1}$$

The heat-flux equation has the following form:

$$\frac{\partial\theta}{\partial t} + v_x\frac{\partial\theta}{\partial x} + v_y\frac{\partial\theta}{\partial y} = \frac{\nu}{P}\frac{\partial^2\theta}{\partial y^2} + \nu\left(1 - \frac{1}{P}\right)\frac{\partial}{\partial y}\left(v_x\frac{\partial v_x}{\partial y}\right) \tag{2}$$

$$\theta = H + {}^1/_2 v_x^2$$

In this equation $H$ is the enthalpy, $P$ the Prandtl number.

We write down Equation (2) in variables $t$, $x$, $\zeta$, using (1), thus:

$$\frac{x}{V_\infty}\frac{\partial\theta}{\partial t} + xf'\frac{\partial\theta}{\partial x} - \frac{f}{2}\frac{\partial\theta}{\partial\zeta} = \frac{1}{P}\frac{\partial^2\theta}{\partial\zeta^2} + V_\infty^2\left(1 - \frac{1}{P}\right)\frac{d}{d\zeta}(f'f'') \tag{3}$$

The boundary conditions and initial conditions for Equation (3) are as follows:

$$\theta(t, x, 0) = H_w(t), \qquad \theta(t, x, \infty) = \theta_{00} = \text{const}, \qquad \theta(0, x, \zeta) = \theta_c$$

In this expression $\theta_c(f')$ is the stagnation enthalpy profile for steady flow with $T_{w_0}$ = const. It will be evident from what follows that the change in enthalpy on the wall can satisfy the more general law

$$H_w(t,x) = a_0(t) + a_1(t)\,x + a_2(t)\,x^2 + \ldots$$

We will assume the solution to be of the form

$$\theta = \theta_c + \theta_1(t, x, \zeta)$$

For $\theta_1$ we arrive at the following homogeneous equation:

$$\frac{x}{V_\infty}\frac{\partial\theta_1}{\partial t} + xf'\,\frac{\partial\theta_1}{\partial x} - \frac{f}{2}\,\frac{d\theta_1}{d\zeta} = \frac{1}{P}\,\frac{\partial^2\theta_1}{\partial\zeta^2} \tag{4}$$

which has the following initial, and boundary, conditions:

$$\theta_1(t, x, 0) = H_w - H_{w_0}, \qquad \theta_1(t, x, \infty) = \theta_1(0, x, \zeta) = 0$$

Now if we write the boundary condition on the wall in the form

$$H_w - H_{w0} = \Sigma A_n t^n$$

we can seek a solution in the form of a series

$$\theta_1 = \Sigma A_n t^n \varphi_n(t, x, \zeta)$$

in which functions $\phi_n(t, x, \zeta)$ satisfy the equation

$$\frac{x}{V_\infty}\left(\frac{\partial\varphi_n}{\partial t} + \frac{n}{t}\,\varphi_n\right) + xf'\,\frac{\partial\varphi_n}{\partial x} - \frac{f}{2}\,\frac{\partial\varphi_n}{\partial\zeta} = \frac{1}{P}\,\frac{\partial^2\varphi_n}{\partial\zeta^2}$$

and conditions

$$\varphi_n(t, x, 0) = 1, \qquad \varphi_n(t, x, \infty) = 0 \tag{5}$$

It follows from dimensional analysis considerations that the functions $\phi_n$ only depend on two dimensionless variables $\xi = x/V_\infty t$ and $\zeta$.

Equation (5) then takes the following form:

$$n\xi\varphi_n - \xi^2\,\frac{\partial\varphi_n}{\partial\xi} + \xi f'\,\frac{\partial\varphi_n}{\partial\xi} - \frac{f}{2}\,\frac{\partial\varphi_n}{\partial\zeta} = \frac{1}{P}\,\frac{\partial^2\varphi_n}{\partial\zeta^2} \tag{6}$$

Series solutions to the latter equation can be obtained for both small and large values of the variable $\xi$.

It is easy to show that for large values of $\xi$ the solution for $\phi_n$ can be represented as follows:

$$\varphi_n = \sum_0^\infty y_k(z)\,\xi^{-3/2k} \qquad (z = \zeta\sqrt{\xi})$$

where functions $y_k(z)$ satisfy the equations and the boundary conditions

$$P^{-1}y_0'' + \tfrac{1}{2}zy_0' - ny_0 = 0$$

$$P^{-1}y_1'' + \tfrac{1}{2}zy_1' - \left(n + \tfrac{3}{2}\right)y_1 - \tfrac{1}{4}\alpha z^2 y_0' = 0$$

$$P^{-1}y_2'' + \tfrac{1}{2}zy_2' - (n+3)y_2 + \tfrac{3}{2}\alpha z y_1 - \tfrac{1}{4}\alpha z^2 y_1' + \tfrac{1}{5!}\alpha^2 z^5 y_0' = 0$$

$$P^{-1}y_3'' + \tfrac{1}{2}zy_3' - \left(n + \tfrac{9}{2}\right)y_3 - \tfrac{3}{4\cdot 4!}\alpha^2 z^4 y_1 + 3\alpha z y_2 - \tfrac{1}{4}\alpha z^2 y_2' + \tfrac{1}{5!}\alpha^2 z^5 y_1' - \tfrac{77}{8\cdot 8!}\alpha^3 z^8 y_0' = 0$$

$$P^{-1}y_4'' + \tfrac{1}{2}zy_4' - (n+6)y_4 + \tfrac{9}{2}\alpha z y_3 - \tfrac{3}{2\cdot 4!}\alpha^2 z^4 y_2 + \tfrac{33}{8!}\alpha^3 z^7 y_1 + \tfrac{1875}{8\cdot 11!}\alpha^4 z^{11} y_0' - \tfrac{77}{8\cdot 8!}\alpha^3 z^8 y_1' + \tfrac{1}{5!}\alpha^2 z^5 y_2' - \tfrac{1}{4}\alpha z^2 y_3' = 0$$

$$\alpha = f''(0), \qquad y_0(0) = 1, \qquad y_0(\infty) = 0, \qquad y_k(0) = y_k(\infty) = 0 \qquad \text{for } k > 0$$

The solution for $y_k(z)$ can be put in the form

$$y_0(z) = c_0 H_{2n}\left(\tfrac{1}{2} iz\sqrt{P}\right) Z_{2n}\left(\tfrac{1}{2} z\sqrt{P}, \infty\right)$$

$$y_k(z) = H_{2n+3k}\left(\tfrac{1}{2} iz\sqrt{P}\right)\int_\infty^z \exp\left(-\tfrac{1}{4}Px^2\right)\left[H_{2n+3k}\left(\tfrac{1}{2} ix\sqrt{P}\right)\right]^{-2} \times \int_0^x \exp\left(\tfrac{1}{4}t^2 P\right) H_{2n+3k}\left(\tfrac{1}{2} it\sqrt{P}\right) h_k(t)\,dt\,dx + c_k H_{2n+3k}\left(\tfrac{1}{2} iz\sqrt{P}\right)\int_\infty^z \exp\left(-\tfrac{1}{4}x^2 P\right)\left[H_{2n+3k}\left(\tfrac{1}{2} ix\sqrt{P}\right)\right]^{-2} dx$$

$$(k = 1, 2, 3, \ldots)$$

Here $H_m$ are Hermite polynomials of degree $m$

$$z_m(\beta, \vartheta) = \int_\vartheta^\beta \frac{e^{-x^2}dx}{H_m^{\,2}(ix)}$$

$$h_1(t) = \tfrac{1}{4}\alpha P t^2 y_0'(t), \quad h_2(t) = \tfrac{1}{4}\alpha P t^2 y_1'(t) - \tfrac{3}{2}\alpha P t y_1 - \tfrac{1}{5!}\alpha^2 P t^5 y_0'(t)$$

$$h_3(t) = \tfrac{1}{4}\alpha P t^2 y_2'(t) + \tfrac{3}{4\cdot 4!}\alpha^2 P t^4 y_1(t) - 3\alpha t P y_2(t) - \tfrac{1}{5!}\alpha^2 P t^5 y_1'(t) + \tfrac{77}{8\cdot 8!}\alpha^3 P t^8 y_0'(t)$$

$$y_4(t) = \frac{1}{4}\alpha P t^2 y_3'(t) - \frac{9}{2}\alpha P t y_3(t) + \frac{3}{2\cdot 4!}\alpha^2 P t^5 y_2(t) -$$
$$- \frac{33}{8!}\alpha^3 P t^7 y_1(t) - \frac{1875}{8\cdot 11!}\alpha^4 P t'' y_0'(t) + \frac{77}{8\cdot 8!}\alpha^3 P t^8 y_1'(t) - \frac{1}{5!}\alpha^2 P t^5 y_2(t)$$

$$c_0 = [H_{2n}(0) z_{2n}(0, \infty)]^{-1} \quad \text{for } 2n \text{ even}$$

$$c_0 = -i\,[H_{2n}'(0)]^{-1} \quad \text{for } 2n \text{ odd}$$

Coefficients $c_k$ are found from the condition $y_k(0) = 0$.

To find the solution for small values of $\xi$ it is desirable, in Equation (6), to transform to the variables $\xi$ and $f'$:

$$n\xi\varphi_n + \xi(f' - \xi)\frac{\partial\varphi_n}{\partial\xi} = \frac{f''^2}{P}\frac{\partial^2\varphi_n}{\partial f'^2} + \left(1 - \frac{1}{P}\right)\frac{ff''}{2}\frac{\partial\varphi_n}{\partial f'} \tag{7}$$

Then, for the function $Y_k(f')$ of the series

$$\varphi_n = \sum_0^\infty \xi^k Y_k(f')$$

we obtain a system of ordinary differential equations of the form

$$\frac{f''^2}{P}\frac{d^2 Y_k}{df'^2} + \left(1 - \frac{1}{P}\right)\frac{ff''}{2}\frac{dY}{df'} = kf' Y_k + (n - k + 1) Y_{k-1} \tag{8}$$

with boundary conditions

$$Y_0(0) = 1, \quad Y_0(1) = 0, \quad Y_k(0) = Y_k(1) = 0 \quad \text{for } k < 0$$

The solution to the equation for $Y_0(f_0')$ corresponds to a quasi-steady temperature variation in the boundary layer; it can be found, for instance in [1].

It is easy to obtain an approximate solution for the subsequent functions $Y_k$ using the method of integral expressions. In such a case it is desirable to represent solutions $Y_k(f')$ as $m$th degree polynomials in $f'$; the latter having to satisfy the boundary conditions

$$Y_k(0) = Y_k(1) = 0 \quad \text{for } k \geqslant 1$$
$$Y_1''(0) = \frac{nP}{\alpha^2}, \quad Y_k''(0) = 0 \quad \text{for } k \geqslant 2$$

and, additionally, $m - 2$ conditions obtained by integrating Equations (8) multiplied by $f'^{l}$ ($l$ = 0, 1, 2, ..., $m$ − 3) from 0 to 1.

By representing $Y_k(f')$ as polynomials we obtain a system of linear algebraic equations for the coefficients.

For instance, for $m$ = 3

$$Y_k = a_k\,(f' - f'^3) \qquad \text{for } k > 1$$

$$Y_1 = a_1 f' + \frac{n}{2\alpha^2 P} f'^2 - \left(a_1 + \frac{n}{2\alpha^2 P}\right) f'^3$$

and from the integral relations for $P$ = 1 and $k \geqslant 3$ we obtain

$$a_k = \frac{(k - 1 - n)}{4\left(\frac{2}{15}k + 6c_1\right)}, \qquad c_1 = \int_0^1 f''^2 f'\,df'$$

From the above, we derive

$$a_k = \frac{(k-1-n)(k-2-n)\ldots(2-n)\,a_2}{4^{k-2}}\left[\prod_{i=3}^{i=k}\left(\frac{2}{15}i + 6c_1\right)\right]^{-1}$$

It is easy to deduce the following:

$$a_1 = \frac{nb_0}{\frac{2}{15} + 6C_1}, \qquad b_0 = \frac{c_0}{\alpha^2} - \frac{3c_1}{\alpha^2} + \frac{1}{40\alpha^2} - \frac{1}{2}, \qquad c_0 = \int_0^1 f''^2 df'$$

$$a_2 = \frac{n(1-n)\,b_1}{4\left(\frac{2}{15} + 6c_1\right)\left(\frac{2\cdot 2}{15} + 6c_1\right)}, \qquad b_1 = b_0 + \frac{1}{6\alpha^2}\left(\frac{2}{15} + 6c_1\right)$$

Therefore

$$a_k = \frac{(k-1-n)(k-2-n)\ldots(1-n)\,nb_1}{4^{k-1}}\left[\prod_{i=1}^{i=k}\left(\frac{2}{15}i + 6c_1\right)\right]^{-1} \text{ for } k > 1$$

The series for $(\partial\phi_n/\partial f')$ converges for $\xi$ < 8/15. If we make use of the expression for $\phi_n$ for small and for large values of $\xi$ we arrive at an approximate solution of the problem over the whole range of variation of this variable.

The solution obtained here can be used for the case where, instead of incompressible fluid, we deal with a gas which obeys a viscosity law $\mu\rho$ = const. It was shown [2] that for this case the variable $\zeta$ should be replaced by

$$\frac{1}{\sqrt{\nu x / V_\infty}} \int_0^y \frac{\rho}{\bar{\rho}}\,dy$$

BIBLIOGRAPHY

1. Howarth, L., Sovremennoe sostoianie aerodinamike bol'shykh skorostei (Modern Development in Fluid Dynamics - Compressible Flow). *Izd. In. Lit.* (Foreign Literature Publications), Moscow, 1958.

2. Demianov, Iu.A., Formirovanie pogranichnogo sloi na plastine za dvizhushchimsia skachkom uplotneiia (Boundary-layer formation on a plate behind a moving shock wave). *PMM* Vol. 21, No. 3, 1957.

3. Sparrow, G., Nonsteady surface-temperature effect on forced convection heat transfer. *IAS,* 10, 1957.

4. Sparrow, The combined effect of the nonsteady velocity of the surface temperature on the heat transfer. *Jet Propulsion* Vol. 28, No. 6, 1958.

*Translated by* V.H.B.